FHSST Authors

The Free High School Science Texts: Textbooks for High School Students Studying the Sciences Mathematics Grades 10 - 12

Version 0
September 17, 2008

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FHSST Core Team

Mark Horner ; Samuel Halliday ; Sarah Blyth ; Rory Adams ; Spencer Wheaton

FHSST Editors

Jaynie Padayachee ; Joanne Boulle ; Diana Mulcahy ; Annette Nell ; René Toerien ; Donovan Whitfield

FHSST Contributors

Rory Adams ; Prashant Arora ; Richard Baxter ; Dr. Sarah Blyth ; Sebastian Bodenstein ; Graeme Broster ; Richard Case ; Brett Cocks ; Tim Crombie ; Dr. Anne Dabrowski ; Laura Daniels ; Sean Dobbs ; Fernando Durrell ; Dr. Dan Dwyer ; Frans van Eeden ; Giovanni Franzoni ; Ingrid von Glehn ; Tamara von Glehn ; Lindsay Glesener ; Dr. Vanessa Godfrey ; Dr. Johan Gonzalez ; Hemant Gopal ; Umeshree Govender ; Heather Gray ; Lynn Greeff ; Dr. Tom Gutierrez ; Brooke Haag ; Kate Hadley ; Dr. Sam Halliday ; Asheena Hanuman ; Neil Hart ; Nicholas Hatcher ; Dr. Mark Horner ; Mfandaidza Hove ; Robert Hovden ; Jennifer Hsieh ; Clare Johnson ; Luke Jordan ; Tana Joseph ; Dr. Jennifer Klay ; Lara Kruger ; Sihle Kubheka ; Andrew Kubik ; Dr. Marco van Leeuwen ; Dr. Anton Machacek ; Dr. Komal Maheshwari ; Kosma von Maltitz ; Nicole Masureik ; John Mathew ; JoEllen McBride ; Nikolai Meures ; Riana Meyer ; Jenny Miller ; Abdul Mirza ; Asogan Moodaly ; Jothi Moodley ; Nolene Naidu ; Tyrone Negus ; Thomas O'Donnell ; Dr. Markus Oldenburg ; Dr. Jaynie Padayachee ; Nicolette Pekeur ; Sirika Pillay ; Jacques Plaut ; Andrea Prinsloo ; Joseph Raimondo ; Sanya Rajani ; Prof. Sergey Rakityansky ; Alastair Ramlakan ; Razvan Remsing ; Max Richter ; Sean Riddle ; Evan Robinson ; Dr. Andrew Rose ; Bianca Ruddy ; Katie Russell ; Duncan Scott ; Helen Seals ; Ian Sherratt ; Roger Sieloff ; Bradley Smith ; Greg Solomon ; Mike Stringer ; Shen Tian ; Robert Torregrosa ; Jimmy Tseng ; Helen Waugh ; Dr. Dawn Webber ; Michelle Wen ; Dr. Alexander Wetzler ; Dr. Spencer Wheaton ; Vivian White ; Dr. Gerald Wigger ; Harry Wiggins ; Wendy Williams ; Julie Wilson ; Andrew Wood ; Emma Wormauld ; Sahal Yacoob ; Jean Youssef

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Chapter 31

Geometry - Grade 11

31.1 Introduction

Activity :: Extension : History of Geometry
Work in pairs or groups and investigate the history of the development of geometry in the last 1500 years. Describe the various stages of development and how different cultures used geometry to improve their lives.

The works of the following people or cultures can be investigated:

1. Islamic geometry (c. 700 - 1500)
   A Thabit ibn Qurra
   B Omar Khayyam
   C Sharafeddin Tusi
2. Geometry in the 17th - 20th centuries (c. 700 - 1500)

31.2 Right Pyramids, Right Cones and Spheres

A pyramid is a geometric solid that has a polygon base and the base is joined to an apex. Examples of pyramids are shown in Figure 31.1.

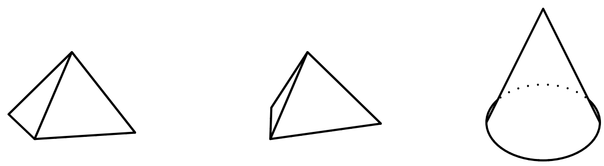


Figure 31.1: Examples of a square pyramid, a triangular pyramid and a cone.

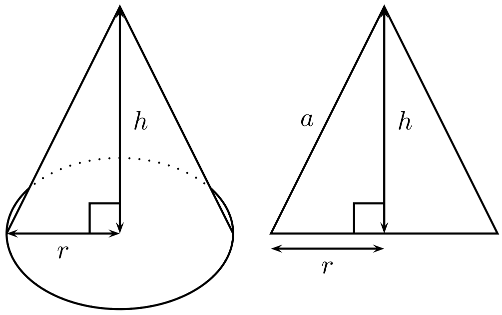
Method: Surface Area of a Pyramid
The surface area of a pyramid is calculated by adding the area of each face together.

Worked Example 127: Surface Area

Question: If a cone has a height of h and a base of radius r, show that the surface area is $\pi r^2 + \pi r\sqrt{r^2 + h^2}$.

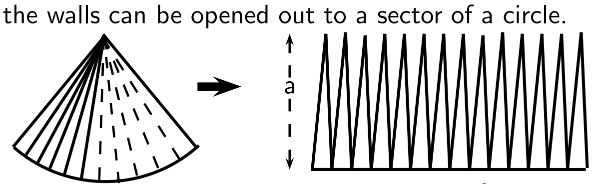
Answer

Step 1 : Draw a picture



Step 2 : Identify the faces that make up the cone

The cone has two faces: the base and the walls. The base is a circle of radius r and the walls can be opened out to a sector of a circle.



2πr = circumference

This curved surface can be cut into many thin triangles with height close to a (a is called a *slant height*). The area of these triangles will add up to $\frac{1}{2}$×base×height which is $\frac{1}{2} \times 2\pi r \times a = \pi r a$

Step 3 : Calculate a

a can be calculated by using the Theorem of Pythagoras. Therefore:

$$a = \sqrt{r^2 + h^2}$$

Step 4 : Calculate the area of the circular base

$$A_b = \pi r^2$$

Step 5 : Calculate the area of the curved walls

$$\begin{aligned} A_w &= \pi r a \\ &= \pi r\sqrt{r^2 + h^2} \end{aligned}$$

Step 6 : Calculate surface area A

$$\begin{aligned} A &= A_b + A_w \\ &= \pi r^2 + \pi r\sqrt{r^2 + h^2} \end{aligned}$$

Method:

Volume of a Pyramid

The volume of a pyramid is found by:

$$V = \frac{1}{3}A \cdot h$$

where A is the area of the base and h is the height.

A cone is a pyramid, so the volume of a cone is given by

$$V = \frac{1}{3}\pi r^2 h.$$

A square pyramid has volume

$$V = \frac{1}{3}a^2 h$$

where a is the side length.

Worked Example 128: Volume of a Pyramid

Question: What is the volume of a square pyramid, 3cm high with a side length of 2cm?

Answer

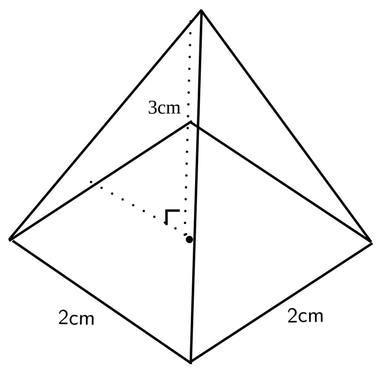
Step 1 : Determine the correct formula

The volume of a pyramid is

$$V = \frac{1}{3}A \cdot h,$$

which for a square base means

$$V = \frac{1}{3}a \cdot a \cdot h.$$



Step 2 : Substitute the given values

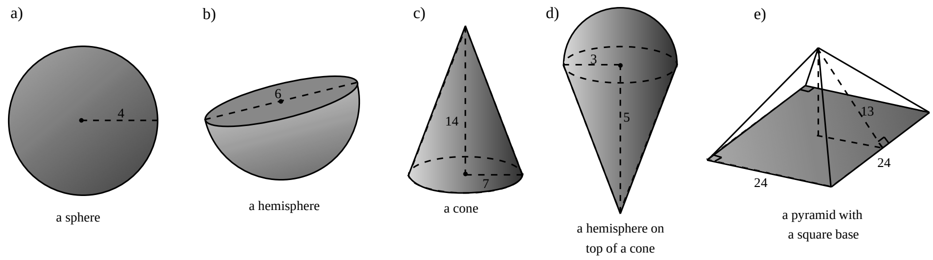
$$\begin{aligned} &= \frac{1}{3} \cdot 2 \cdot 2 \cdot 3 \\ &= \frac{1}{3} \cdot 12 \\ &= 4 \text{ cm}^3 \end{aligned}$$

We accept the following formulae for volume and surface area of a sphere (ball).

$$\begin{aligned} \text{Surface area} &= 4\pi r^2 \\ \text{Volume} &= \frac{4}{3}\pi r^3 \end{aligned}$$

Exercise: Surface Area and Volume

1. Calculate the volumes and surface areas of the following solids: *Hint for (e): find the perpendicular height using Pythagoras.



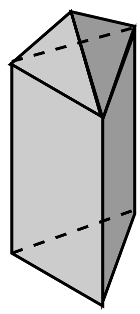
2. Water covers approximately 71% of the Earth's surface. Taking the radius of the Earth to be 6378 km, what is the total area of land (area not covered by water)?

3. A right triangular pyramid is placed on top of a right triangular prism. The prism has an equilateral triangle of side length 20 cm as a base, and has a height of 42 cm. The pyramid has a height of 12 cm.

   A Find the total volume of the object.

   B Find the area of each face of the pyramid.

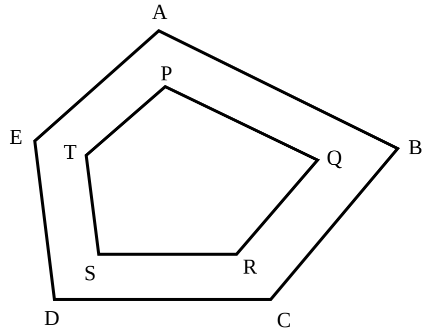
   C Find the total surface area of the object.



31.3 Similarity of Polygons

In order for two polygons to be similar the following must be true:

1. All corresponding angles must be congruent.
2. All corresponding sides must be in the same proportion to each other.



If

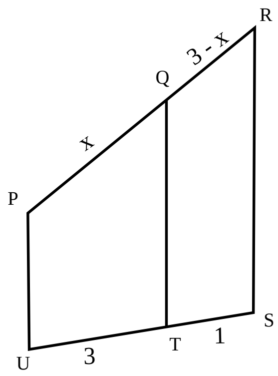
1. $\hat{A} = \hat{P}$; $\hat{B} = \hat{Q}$; $\hat{C} = \hat{R}$; $\hat{D} = \hat{S}$; $\hat{E} = \hat{T}$
   and
2. $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CD}{RS} = \frac{DE}{ST} = \frac{EA}{TP}$

then the polygons ABCDE and PQRST are similar.

Worked Example 129: Similarity of Polygons

Question:

Polygons PQTU and PRSU are similar.
Find the value of x.



Answer

Step 1 : Identify corresponding sides

Since the polygons are similar,

$$\begin{aligned} \frac{PQ}{PR} &= \frac{TU}{SU} \\ \therefore \frac{x}{x+(3-x)} &= \frac{3}{1} \\ \therefore \frac{x}{3} &= 3 \\ \therefore x &= 9 \end{aligned}$$

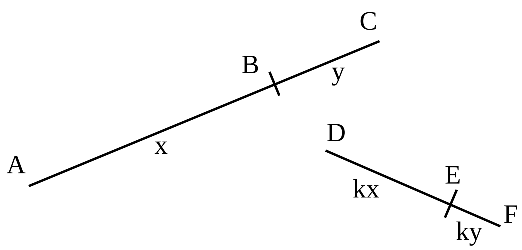
31.4 Triangle Geometry

31.4.1 Proportion

Two line segments are divided in the same proportion if the ratios between their parts are equal.

$$\frac{AB}{BC} = \frac{x}{y} = \frac{kx}{ky} = \frac{DE}{EF}$$

$\therefore$ the line segments are in the same proportion

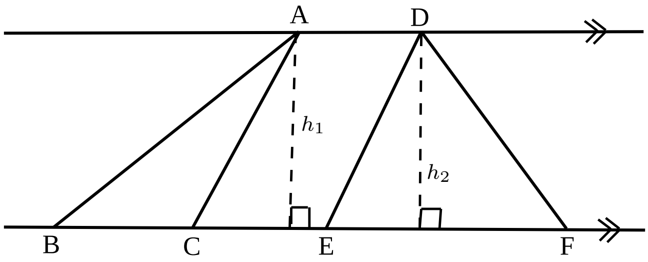


If the line segments are proportional, the following also hold

1. $AC \cdot FE = CB \cdot DF$
2. $\frac{CB}{AC} = \frac{FE}{DF}$
3. $\frac{AB}{BC} = \frac{DE}{FE}$ and $\frac{BC}{AB} = \frac{FE}{DE}$
4. $\frac{AB}{AC} = \frac{DE}{DF}$ and $\frac{AC}{AB} = \frac{DF}{DE}$

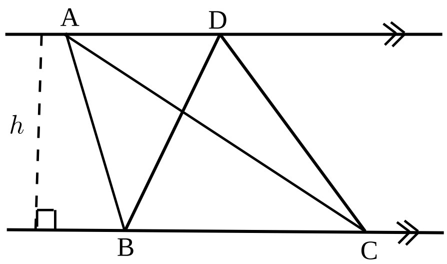
- Triangles with equal heights have areas which are in the same proportion to each other as the bases of the triangles.

$$\begin{aligned} h_1 &= h_2 \\ \therefore \frac{\text{area } \triangle ABC}{\text{area } \triangle DEF} &= \frac{\frac{1}{2}BC \times h_1}{\frac{1}{2}EF \times h_2} = \frac{BC}{EF} \end{aligned}$$



- A special case of this happens when the bases of the triangles are equal:
  Triangles with equal bases between the same parallel lines have the same area.

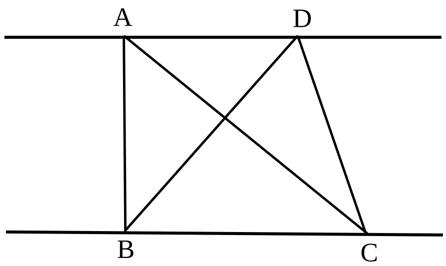
$$\text{area } \triangle ABC = \frac{1}{2} \cdot h \cdot BC = \text{ area } \triangle DBC$$



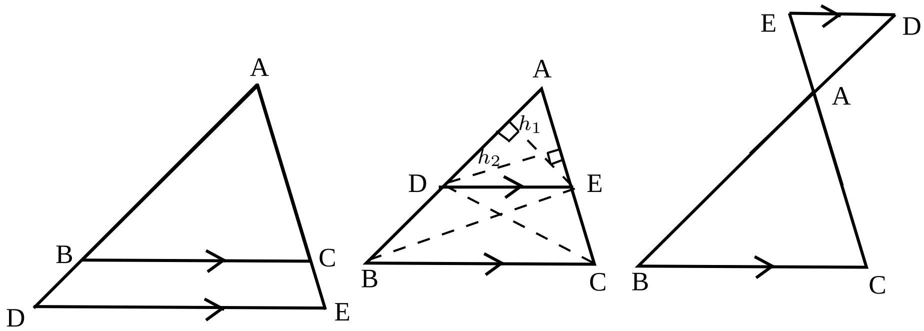
- Triangles on the same side of the same base, with equal areas, lie between parallel lines.

If area △ ABC = area △ BDC,

then AD ∥ BC.



Theorem 1. *Proportion Theorem:A line drawn parallel to one side of a triangle divides the other two sides proportionally.*



Given:△ABC with line DE ∥ BC

R.T.P.:

$$\frac{AD}{DB} = \frac{AE}{EC}$$

Proof:

Draw h_1 from E perpendicular to AD, and h_2 from D perpendicular to AE.
Draw BE and CD.

$$\begin{aligned}
\frac{\text{area } \triangle\text{ADE}}{\text{area } \triangle\text{BDE}} &= \frac{\frac{1}{2}AD \cdot h_1}{\frac{1}{2}DB \cdot h_1} = \frac{AD}{DB} \\
\frac{\text{area } \triangle\text{ADE}}{\text{area } \triangle\text{CED}} &= \frac{\frac{1}{2}AE \cdot h_2}{\frac{1}{2}EC \cdot h_2} = \frac{AE}{EC} \\
\text{but area } \triangle\text{BDE} &= \text{area } \triangle\text{CED (equal base and height)} \\
\therefore \frac{\text{area } \triangle\text{ADE}}{\text{area } \triangle\text{BDE}} &= \frac{\text{area } \triangle\text{ADE}}{\text{area } \triangle\text{CED}} \\
\therefore \frac{AD}{DB} &= \frac{AE}{EC}
\end{aligned}$$

∴ DE divides AB and AC proportionally.

Similarly,

$$\begin{aligned}
\frac{AD}{AB} &= \frac{AE}{AC} \\
\frac{AB}{BD} &= \frac{AC}{CE}
\end{aligned}$$

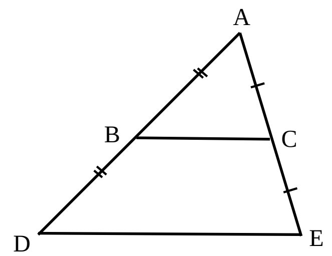
Following from Theorem 1, we can prove the midpoint theorem.:

Theorem 2. *Midpoint Theorem: A line joining the midpoints of two sides of a triangle is parallel to the third side and equal to half the length of the third side.*

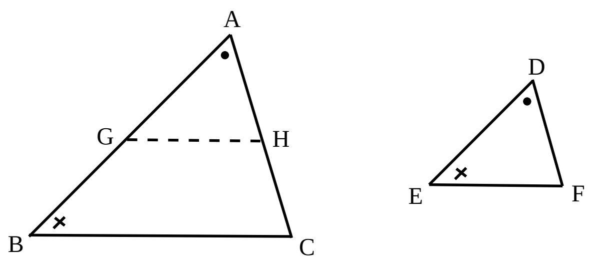
Proof:

This is a special case of the Proportionality Theorem (Theorem 1).

If AB = BD and AC = AE,
then DE ∥ BC and BC = 2DE.



Theorem 3. *Similarity Theorem 1:Equiangular triangles have their sides in proportion and are therefore similar.*



Given:△ABC and △DEF with $\hat{A} = \hat{D}$; $\hat{B} = \hat{E}$; $\hat{C} = \hat{F}$

R.T.P.:

$$\frac{AB}{DE} = \frac{AC}{DF}$$

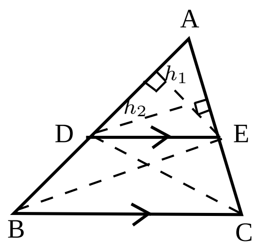
Construct: G on AB, so that AG = DE
H on AC, so that AH = DF

Proof: In △'s AGH and DEF

$$\begin{array}{lll} & AG = DE; AH = DF & \text{(const.)} \\ & \hat{A} = \hat{D} & \text{(given)} \\ \therefore & \triangle AGH \equiv \triangle DEF & \text{(SAS)} \\ \therefore & A\hat{G}H = \hat{E} = \hat{B} & \\ \therefore & GH \parallel BC & \text{(corres. } \angle\text{'s equal)} \\ \therefore & \frac{AG}{AB} = \frac{AH}{AC} & \text{(proportion theorem)} \\ \therefore & \frac{DE}{AB} = \frac{DF}{AC} & \text{(AG = DE; AH = DF)} \\ \therefore & \triangle ABC \,|||\, \triangle DEF & \end{array}$$

Important: ||| means "is similar to"

Theorem 4. *Similarity Theorem 2:Triangles with sides in proportion are equiangular and therefore similar.*



Given:△ABC with line DE such that

$$\frac{AD}{DB} = \frac{AE}{EC}$$

R.T.P.:$DE \parallel BC$; △ADE ||| △ABC

Proof:

Draw h_1 from E perpendicular to AD, and h_2 from D perpendicular to AE.
Draw BE and CD.

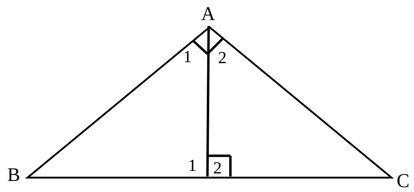
$$\begin{aligned}
\frac{\text{area } \triangle\text{ADE}}{\text{area } \triangle\text{BDE}} &= \frac{\frac{1}{2}AD\cdot h_1}{\frac{1}{2}DB\cdot h_1} = \frac{AD}{DB} \\
\frac{\text{area } \triangle\text{ADE}}{\text{area } \triangle\text{CED}} &= \frac{\frac{1}{2}AE\cdot h_2}{\frac{1}{2}EC\cdot h_2} = \frac{AE}{EC} \\
\text{but } \frac{AD}{DB} &= \frac{AE}{EC} \text{ (given)} \\
\therefore \frac{\text{area } \triangle\text{ADE}}{\text{area } \triangle\text{BDE}} &= \frac{\text{area } \triangle\text{ADE}}{\text{area } \triangle\text{CED}} \\
\therefore \text{area } \triangle\text{BDE} &= \text{area } \triangle\text{CED} \\
\therefore DE \parallel BC & \quad \text{(same side of equal base DE, same area)} \\
\therefore A\hat{D}E &= A\hat{B}C \text{ (corres } \angle\text{'s)} \\
\text{and } A\hat{E}D &= A\hat{C}B
\end{aligned}$$

$$\therefore \triangle\text{ADE and } \triangle\text{ABC are equiangular}$$

$$\therefore \triangle ADE \;|||\; \triangle ABC \text{ (AAA)}$$

Theorem 5. *Pythagoras' Theorem:The square on the hypotenuse of a right angled triangle is equal to the sum of the squares on the other two sides.*

Given:△ ABC with $\hat{A} = 90°$



R.T.P.:$BC^2 = AB^2 + AC^2$

Proof:

$$\begin{aligned}
\text{Let } \hat{C} &= x \\
\therefore \hat{A}_2 &= 90° - x \text{ } (\angle \text{ 's of a } \triangle \text{)} \\
\therefore \hat{A}_1 &= x \\
\hat{B} &= 90° - x \text{ } (\angle \text{ 's of a } \triangle \text{)} \\
\hat{D}_1 &= \hat{D}_2 = \hat{A} = 90°
\end{aligned}$$

$$\therefore \triangle\text{ABD } |||\triangle\text{CBA and } \triangle\text{CAD } |||\triangle \text{ CBA (AAA)}$$

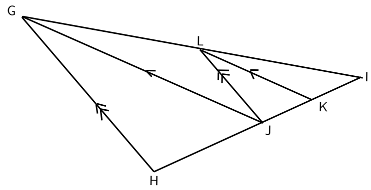
$$\therefore \frac{AB}{CB} = \frac{BD}{BA} = \left(\frac{AD}{CA}\right) \text{ and } \frac{CA}{CB} = \frac{CD}{CA} = \left(\frac{AD}{BA}\right)$$

$$\therefore AB^2 = CB \times BD \text{ and } AC^2 = CB \times CD$$

$$\begin{aligned}
\therefore AB^2 + AC^2 &= CB(BD + CD) \\
&= CB(CB) \\
&= CB^2 \\
\text{i.e. } BC^2 &= AB^2 + AC^2
\end{aligned}$$

Worked Example 130: Triangle Geometry 1

Question: In △ GHI, GH ∥ LJ; GJ ∥ LK and $\frac{JK}{KI} = \frac{5}{3}$. Determine $\frac{HJ}{KI}$.



Answer

Step 1 : Identify similar triangles

$$\begin{aligned} L\hat{I}J &= G\hat{I}H \\ J\hat{L}I &= H\hat{G}I && \text{(Corres. } \angle\text{s)} \\ \therefore \triangle LIJ &\ ||| \ \triangle GIH && \text{(Equiangular } \triangle\text{s)} \end{aligned}$$

$$\begin{aligned} L\hat{I}K &= G\hat{I}J \\ K\hat{L}I &= J\hat{G}I && \text{(Corres. } \angle\text{s)} \\ \therefore \triangle LIK &\ ||| \ \triangle GIJ && \text{(Equiangular } \triangle\text{s)} \end{aligned}$$

Step 2 : Use proportional sides

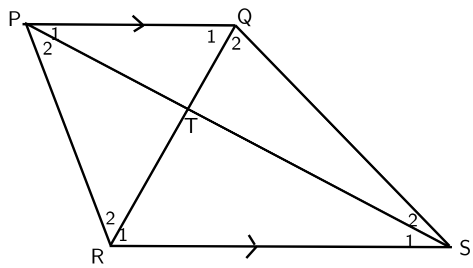
$$\begin{aligned} \frac{HJ}{JI} &= \frac{GL}{LI} && (\triangle LIJ \ ||| \ \triangle GIH) \\ \text{and } \frac{GL}{LI} &= \frac{JK}{KI} && (\triangle LIK \ ||| \ \triangle GIJ) \\ &= \frac{5}{3} \\ \therefore \frac{HJ}{JI} &= \frac{5}{3} \end{aligned}$$

Step 3 : Rearrange to find the required ratio

$$\begin{aligned} \frac{HJ}{KI} &= \frac{HJ}{JI} \times \frac{JI}{KI} \\ &= \frac{5}{3} \times \frac{8}{3} \\ &= \frac{40}{9} \end{aligned}$$

Worked Example 131: Triangle Geometry 2

Question: PQRS is a trapezium, with PQ ∥ RS. Prove that PT · TR = ST · TQ.



Answer

Step 1 : Identify similar triangles

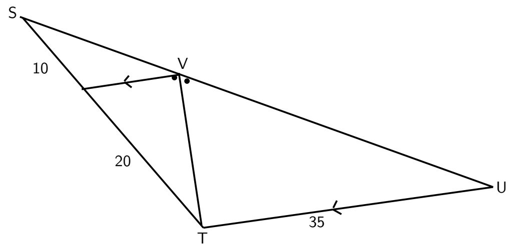
$$\begin{aligned} \hat{P}_1 &= \hat{S}_1 && \text{(Alt. } \angle\text{s)} \\ \hat{Q}_1 &= \hat{R}_1 && \text{(Alt. } \angle\text{s)} \\ \therefore \triangle PTQ &\ ||| \ \triangle STR && \text{(Equiangular } \triangle\text{s)} \end{aligned}$$

Step 2 : Use proportional sides

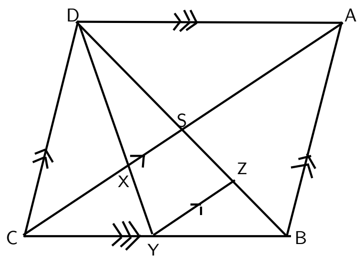
$$\begin{aligned} \frac{PT}{TQ} &= \frac{ST}{TR} && (\triangle \text{ PTQ } ||| \triangle \text{ STR}) \\ \therefore PT \cdot TR &= ST \cdot TQ \end{aligned}$$

Exercise: Triangle Geometry

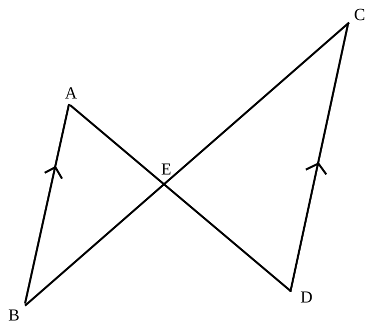
1. Calculate SV



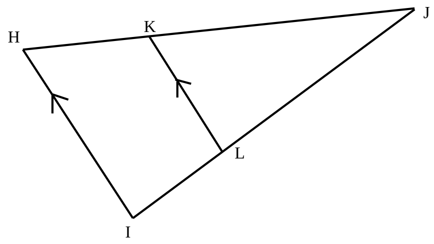
2. $\frac{CB}{YB} = \frac{2}{3}$. Find $\frac{DS}{SB}$.



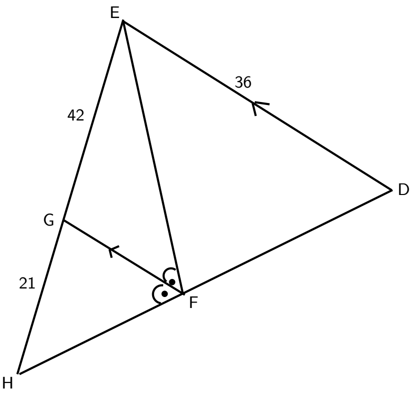
3. Given the following figure with the following lengths, find AE, EC and BE. BC = 15 cm, AB = 4 cm, CD = 18 cm, and ED = 9 cm.



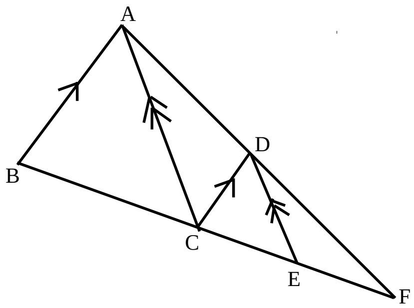
4. Using the following figure and lengths, find IJ and KJ. HI = 26 m, KL = 13 m, JL = 9 m and HJ = 32 m.



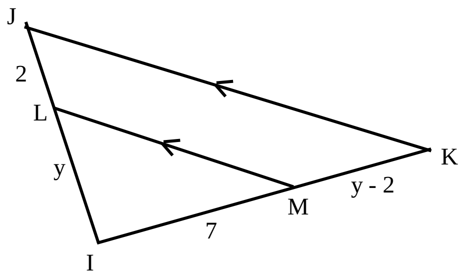
5. Find FH in the following figure.



6. BF = 25 m, AB = 13 m, AD = 9 m, DF = 18m.
Calculate the lengths of BC, CF, CD, CE and EF, and find the ratio $\frac{DE}{AC}$.



7. If LM $\parallel$ JK, calculate y.



31.5 Co-ordinate Geometry

31.5.1 Equation of a Line between Two Points

There are many different methods of specifying the requirements for determining the equation of a straight line. One option is to find the equation of a straight line, when two points are given.

Assume that the two points are $(x_1; y_1)$ and $(x_2; y_2)$, and we know that the general form of the equation for a straight line is:

$$\boxed{y = mx + c} \tag{31.1}$$

So, to determine the equation of the line passing through our two points, we need to determine values for m (the gradient of the line) and c (the y-intercept of the line). The resulting equation is

$$\boxed{y - y_1 = m(x - x_1)} \tag{31.2}$$

where $(x_1; y_1)$ are the co-ordinates of either given point.

Extension: Finding the second equation for a straight line

This is an example of a set of simultaneous equations, because we can write:

$$\begin{aligned} y_1 &= mx_1 + c && (31.3) \\ y_2 &= mx_2 + c && (31.4) \end{aligned}$$

We now have two equations, with two unknowns, m and c.

$$\begin{aligned} \text{Subtract (31.3) from (31.4)} \quad y_2 - y_1 &= mx_2 - mx_1 && (31.5) \\ \therefore \quad m &= \frac{y_2 - y_1}{x_2 - x_1} && (31.6) \\ \text{Re-arrange (31.3) to obtain } c \quad y_1 &= mx_1 + c && (31.7) \\ c &= y_1 - mx_1 && (31.8) \end{aligned}$$

Now, to make things a bit easier to remember, substitute (31.7) into (31.1):

$$\begin{aligned} y &= mx + c && (31.9) \\ &= mx + (y_1 - mx_1) && (31.10) \\ \text{which can be re-arranged to:} \quad y - y_1 &= m(x - x_1) && (31.11) \end{aligned}$$

Important: If you are asked to calculate the equation of a line passing through two points, use:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

to calculate m and then use:

$$y - y_1 = m(x - x_1)$$

to determine the equation.

For example, the equation of the straight line passing through $(-1; 1)$ and $(2; 2)$ is given by first calculating m

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{2 - 1}{2 - (-1)} \\ &= \frac{1}{3} \end{aligned}$$

and then substituting this value into

$$y - y_1 = m(x - x_1)$$

to obtain

$$y - y_1 = \frac{1}{3}(x - x_1).$$

Then substitute $(-1;1)$ to obtain

$$\begin{aligned} y-(1) &= \frac{1}{3}(x-(-1)) \\ y-1 &= \frac{1}{3}x+\frac{1}{3} \\ y &= \frac{1}{3}x+\frac{1}{3}+1 \\ y &= \frac{1}{3}x+\frac{4}{3} \end{aligned}$$

So, $y = \frac{1}{3}x + \frac{4}{3}$ passes through $(-1;1)$ and $(2;2)$.

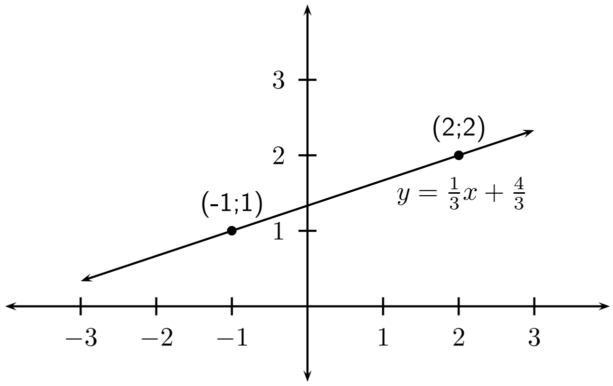


Figure 31.2: The equation of the line passing through $(-1;1)$ and $(2;2)$ is $y = \frac{1}{3}x + \frac{4}{3}$.

Worked Example 132: Equation of Straight Line

Question: Find the equation of the straight line passing through $(-3;2)$ and $(5;8)$.

Answer

Step 1 : Label the points

$$\begin{aligned} (x_1;y_1) &= (-3;2) \\ (x_2;y_2) &= (5;8) \end{aligned}$$

Step 2 : Calculate the gradient

$$\begin{aligned} m &= \frac{y_2-y_1}{x_2-x_1} \\ &= \frac{8-2}{5-(-3)} \\ &= \frac{6}{5+3} \\ &= \frac{6}{8} \\ &= \frac{3}{4} \end{aligned}$$

Step 3 : Determine the equation of the line

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (2) &= \frac{3}{4}(x - (-3)) \\ y &= \frac{3}{4}(x + 3) + 2 \\ &= \frac{3}{4}x + \frac{3}{4} \cdot 3 + 2 \\ &= \frac{3}{4}x + \frac{9}{4} + \frac{8}{4} \\ &= \frac{3}{4}x + \frac{17}{4} \end{aligned}$$

Step 4 : Write the final answer

The equation of the straight line that passes through $(-3; 2)$ and $(5; 8)$ is $y = \frac{3}{4}x + \frac{17}{4}$.

31.5.2 Equation of a Line through One Point and Parallel or Perpendicular to Another Line

Another method of determining the equation of a straight-line is to be given one point, $(x_1; y_1)$, and to be told that the line is parallel or perpendicular to another line. If the equation of the unknown line is $y = mx + c$ and the equation of the second line is $y = m_0x + c_0$, then we know the following:

$$\text{If the lines are parallel, then} \quad m = m_0 \quad (31.12)$$

$$\text{If the lines are perpendicular, then} \quad m \times m_0 = -1 \quad (31.13)$$

Once we have determined a value for m, we can then use the given point together with:

$$y - y_1 = m(x - x_1)$$

to determine the equation of the line.

For example, find the equation of the line that is parallel to $y = 2x - 1$ and that passes through $(-1; 1)$.

First we determine m. Since the line we are looking for is parallel to $y = 2x - 1$,

$$m = 2$$

The equation is found by substituting m and $(-1; 1)$ into:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= 2(x - (-1) \\ y - 1 &= 2(x + 1) \\ y - 1 &= 2x + 2 \\ y &= 2x + 2 + 1 \\ y &= 2x + 3 \end{aligned}$$

31.5.3 Inclination of a Line

In Figure 31.4(a), we see that the line makes an angle θ with the x-axis. This angle is known as the *inclination* of the line and it is sometimes interesting to know what the value of θ is.

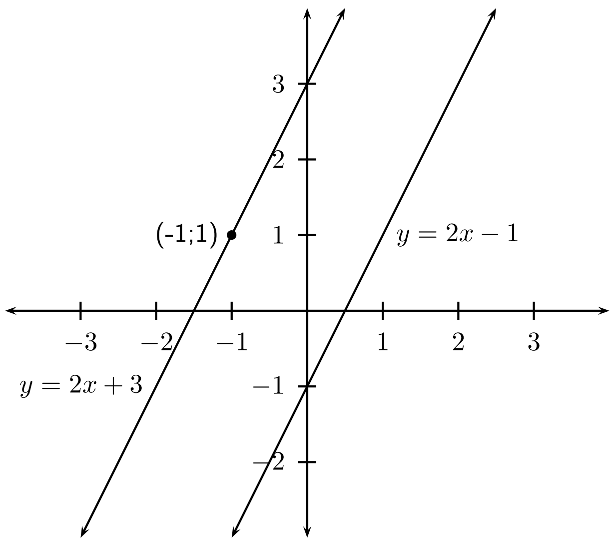


Figure 31.3: The equation of the line passing through $(-1;1)$ and parallel to $y = 2x - 1$ is $y = 2x + 3$. It can be seen that the lines are parallel to each other. You can test this by using your ruler and measuring the distance between the lines at different points.

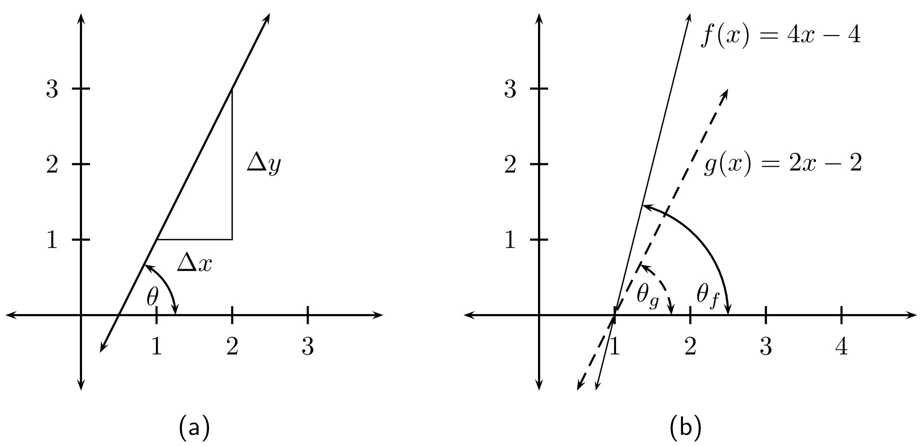


Figure 31.4: (a) A line makes an angle θ with the x-axis. (b) The angle is dependent on the gradient. If the gradient of f is m_f and the gradient of g is m_g then $m_f > m_g$ and $\theta_f > \theta_g$.

Firstly, we note that if the gradient changes, then the value of θ changes (Figure 31.4(b)), so we suspect that the inclination of a line is related to the gradient. We know that the gradient is a ratio of a change in the y-direction to a change in the x-direction.

$$m = \frac{\Delta y}{\Delta x}$$

But, in Figure 31.4(a) we see that

$$\begin{aligned} \tan\theta &= \frac{\Delta y}{\Delta x} \\ \therefore m &= \tan\theta \end{aligned}$$

For example, to find the inclination of the line $y = x$, we know $m = 1$

$$\begin{aligned} \therefore \tan\theta &= 1 \\ \therefore \theta &= 45^\circ \end{aligned}$$

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Exercise: Co-ordinate Geometry

1. Find the equations of the following lines
   A through points $(-1;3)$ and $(1;4)$
   B through points $(7;-3)$ and $(0;4)$
   C parallel to $y = \frac{1}{2}x + 3$ passing through $(-1;3)$
   D perpendicular to $y = -\frac{1}{2}x + 3$ passing through $(-1;2)$
   E perpendicular to $2y + x = 6$ passing through the origin
2. Find the inclination of the following lines
   A $y = 2x - 3$
   B $y = \frac{1}{3}x - 7$
   C $4y = 3x + 8$
   D $y = -\frac{2}{3}x + 3$ (Hint: if m is negative θ must be in the second quadrant)
   E $3y + x - 3 = 0$
3. Show that the line $y = k$ for any constant k is parallel to the x-axis. (Hint: Show that the inclination of this line is 0°.)
4. Show that the line $x = k$ for any constant k is parallel to the y-axis. (Hint: Show that the inclination of this line is 90°.)

31.6 Transformations

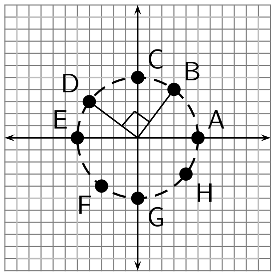
31.6.1 Rotation of a Point

When something is moved around a fixed point, we say that it is *rotated*. What happens to the coordinates of a point that is rotated by 90° or 180° around the origin?

Activity :: Investigation : Rotation of a Point by 90°

Complete the table, by filling in the coordinates of the points shown in the figure.

Point	x-coordinate	y-coordinate
A		
B		
C		
D		
E		
F		
G		
H		

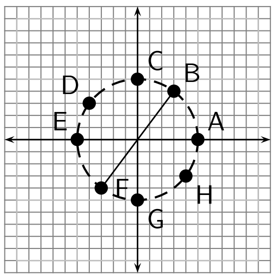


What do you notice about the x-coordinates? What do you notice about the y-coordinates? What would happen to the coordinates of point A, if it was rotated to the position of point C? What about point B rotated to the position of D?

Activity :: Investigation : Rotation of a Point by $180°$

Complete the table, by filling in the coordinates of the points shown in the figure.

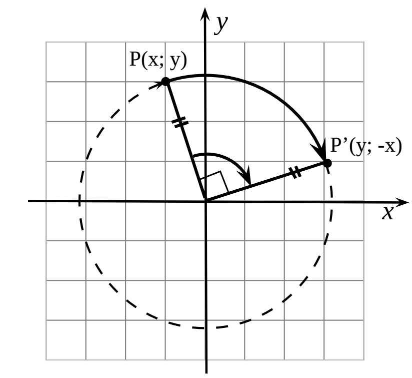
Point	x-coordinate	y-coordinate
A		
B		
C		
D		
E		
F		
G		
H		



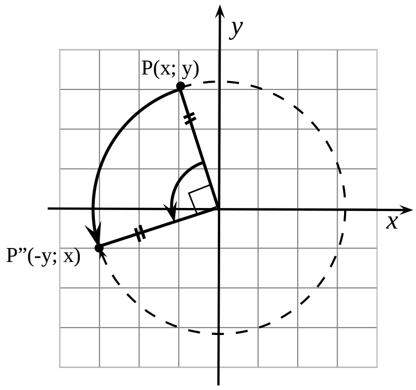
What do you notice about the x-coordinates? What do you notice about the y-coordinates? What would happen to the coordinates of point A, if it was rotated to the position of point E? What about point F rotated to the position of B?

From these activities you should have come to the following conclusions:

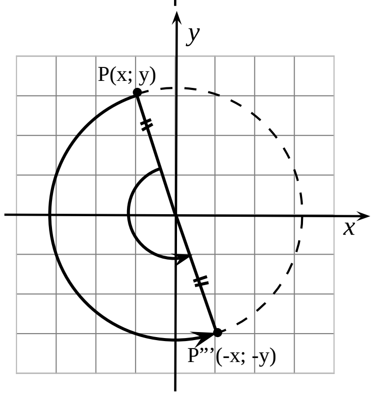
- 90° clockwise rotation:
  The image of a point P$(x;y)$ rotated clockwise through 90° around the origin is P'$(y;-x)$.
  We write the rotation as $(x;y) \rightarrow (y;-x)$.



- 90° anticlockwise rotation:
  The image of a point P$(x;y)$ rotated anticlockwise through 90° around the origin is P'$(-y;x)$.
  We write the rotation as $(x;y) \rightarrow (-y;x)$.



- 180° rotation:
  The image of a point P$(x;y)$ rotated through 180° around the origin is P'$(-x;-y)$.
  We write the rotation as $(x;y) \rightarrow (-x;-y)$.



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Exercise: Rotation

1. For each of the following rotations about the origin:
   (i) Write down the rule.
   (ii) Draw a diagram showing the direction of rotation.

   A OA is rotated to OA′ with A(4;2) and A′(-2;4)

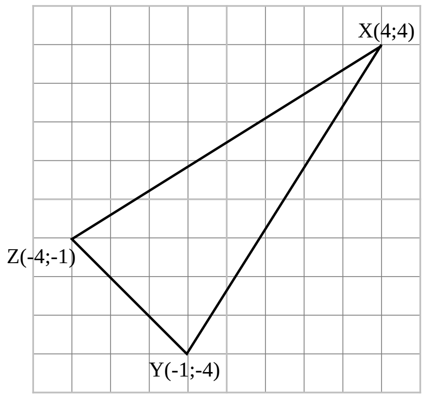
   B OB is rotated to OB′ with B(-2;5) and B′(5;2)

   C OC is rotated to OC′ with C(-1;-4) and C′(1;4)

2. Copy ΔXYZ onto squared paper. The co-ordinates are given on the picture.

   A Rotate ΔXYZ anti-clockwise through an angle of 90° about the origin to give ΔX′Y′Z′. Give the co-ordinates of X′, Y′ and Z′.

   B Rotate ΔXYZ through 180° about the origin to give ΔX′′Y′′Z′′. Give the co-ordinates of X′′, Y′′ and Z′′.



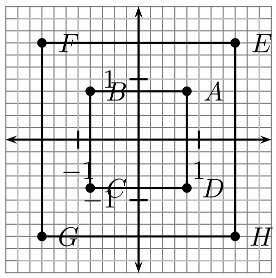
31.6.2 Enlargement of a Polygon 1

When something is made larger, we say that it is *enlarged*. What happens to the coordinates of a polygon that is enlarged by a factor k?

Activity :: Investigation : Enlargement of a Polygon

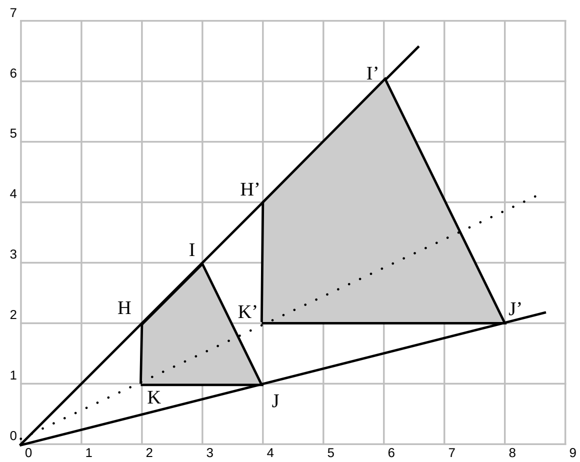
Complete the table, by filling in the coordinates of the points shown in the figure.

Point	x-coordinate	y-coordinate
A		
B		
C		
D		
E		
F		
G		
H		



What do you notice about the x-coordinates?
What do you notice about the y-coordinates?
What would happen to the coordinates of point A, if the square ABCD was enlarged by a factor 2?

Activity :: Investigation : Enlargement of a Polygon 2



In the figure quadrilateral HIJK has been enlarged by a factor of 2 through the origin to become H'I'J'K'. Complete the following table.

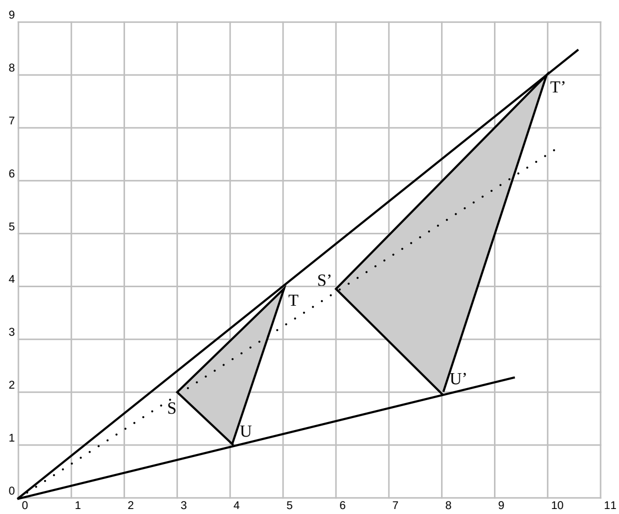
Co-ordinate	Co-ordinate'	Length	Length'
H = (;)	H' = (;)	OH =	OH' =
I = (;)	I' = (;)	OI =	OI' =
J = (;)	J' = (;)	OJ =	OJ' =
K = (;)	K' + (;)	OK =	OK' =

What conclusions can you draw about

1. the co-ordinates
2. the lengths when we enlarge by a factor of 2?

We conclude as follows:
Let the vertices of a triangle have co-ordinates S$(x_1; y_1)$, T$(x_2; y_2)$, U$(x_3; y_3)$. △S'T'U' is an enlargement through the origin of △STU by a factor of c ($c > 0$).

- △STU is a reduction of △S'T'U' by a factor of c.
- △S'T'U' can alternatively be seen as an reduction through the origin of △STU by a factor of $\frac{1}{c}$. (Note that a reduction by $\frac{1}{c}$ is the same as an enlargement by c).
- The vertices of △S'T'U' are S'$(cx_1; cy_1)$, T'(cx_2,cy_2), U'(cx_3,cy_3).
- The distances from the origin are OS' = cOS, OT' = cOT and OU' = cOU.



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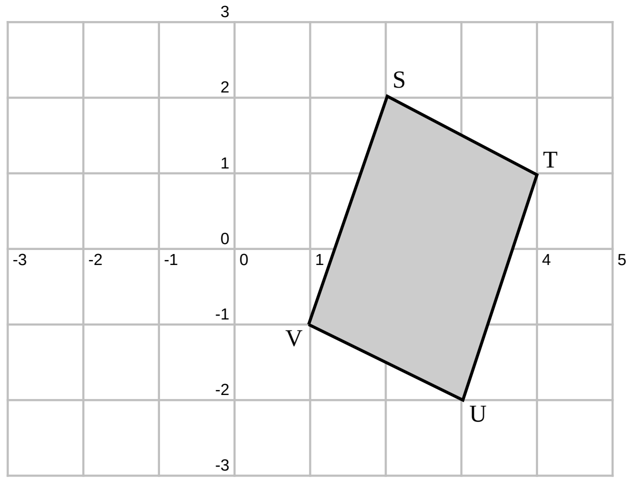
Exercise: Transformations

1. 1) Copy polygon STUV onto squared paper and then answer the following questions.



   A What are the co-ordinates of polygon STUV?

   B Enlarge the polygon through the origin by a constant factor of $c = 2$. Draw this on the same grid. Label it S'T'U'V'.

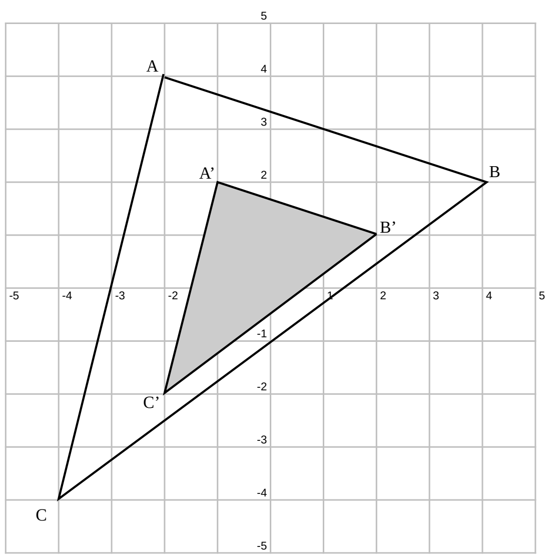
   C What are the co-ordinates of the vertices of S'T'U'V'?

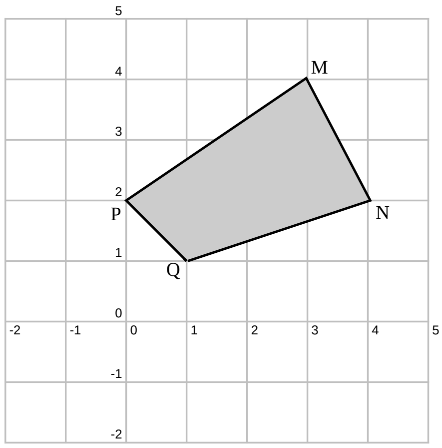
2. △ABC is an enlargement of △A'B'C' by a constant factor of k through the origin.

   A What are the co-ordinates of the vertices of △ABC and △A'B'C'?

   B Giving reasons, calculate the value of k.

   C If the area of △ABC is m times the area of △A'B'C', what is m?



3. 

A What are the co-ordinates of the vertices of polygon MNPQ?

B Enlarge the polygon through the origin by using a constant factor of $c = 3$, obtaining polygon M'N'P'Q'. Draw this on the same set of axes.

C What are the co-ordinates of the new vertices?

D Now draw M"N"P"Q" which is an anticlockwise rotation of MNPQ by 90° around the origin.

E Find the inclination of OM".

Appendix A

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